Equilibrium Points of an AND-OR Tree: under Constraints on Probability

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Abstract

Let d be a prob. distribution on a uniform binary AND-OR tree.

Liu and Tanaka (2007)

If d achieves the equilibrium among ID then d is an IID.

Our result

Given a real number r $(0 < r < 1)$, we consider a constraint $p(:=$ prob[the root has the value 0]$) = r$.
When we restrict ourselves to IDs satisfying this constraint, the above result still holds.

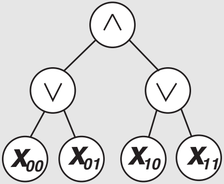
Keys to the solution. (In an IID on an OR-AND tree)
(1) cost / p is a decreasing func. of the probability x of the leaf.
(2) cost′ / p' is a decreasing function of x, too.

Outline

Our setting (1)

Fix a uniform binary AND-OR tree

Example:
height = 2



- $\wedge$ = AND = Min.
- $\vee$ = OR = Max.
- Find: root = 1 (TRUE) or 0 (FALSE)?
- Each leaf is hidden.
- Cost := How many leaves probed?
- Allowed to skip a leaf (α-β pruning).

Our setting (1)

Definition. Alpha-Beta Pruning Algorithm

- Depth-first.
- A child of an AND-gate has the value 0

  $\Downarrow$

  Recognize that the AND-gate has the value 0 without probing the other child (an alpha-cut).
- Similar rule applies to an OR-gate (a beta-cut).

Knuth, D.E. and Moore, R.W.: An analysis of alpha-beta pruning. *Artif. Intell.*, **6** pp. 293–326 (1975).

Our setting (2)

An algorithm is not randomized. Terminal values are randomized.

- An algorithm is assumed to be a deterministic alpha-beta pruning algorithm. It does not toss a coin.
- A (biased) coin is assigned to each leaf. We investigate an ID (indpendent distrib.) on the terminal values.

Preceding study

Theorem A (Liu and Tanaka, 2007)

If d achieves the equilibrium among IDs then d is an IID.

Their proof: "It is not hard."

Here, ID denotes an independent distribution.
IID denotes an independent identical distribution.

Liu, C.-G. and Tanaka, K.:
Eigen-distribution on random assignments for game trees.
Inform. Process. Lett., **104** pp.73–77 (2007).

Preceding study

Equilibrium

"d achieves the equilibrium among ▲▲▲"
$\Leftrightarrow$ d has the property ▲▲▲ and

$$\min_{A_D} \mathrm{cost}(A_D, d) = \max_{\delta} \min_{A_D} \mathrm{cost}(A_D, \delta)$$

Here,
A_D runs over all deterministic alpha-beta pruning algorithms.
δ runs over all prob. distributions s.t. ▲▲▲.

▲▲▲ is e.g., ID.

Preceding study

Theorem A (Liu and Tanaka, 2007)

If d achieves the equilibrium among IDs then d is an IID.

Their proof: "It is not hard."

Is it (↑) really easy to prove?
No. A brutal induction does not work.

We show a stronger form of Theorem A with clever tricks of induction.

Keys to the solution

Lemma 1 (of ours)

Consider an IID on an OR-AND tree.
$x :=$ prob. of a leaf (having the value 0).
$p(x) :=$ prob. of the root (having the value 0).
$c(x) :=$ expected cost of the root.

Then, both of the followings are decreasing functions of x $(0 < x < 1)$.

$$\frac{c(x)}{p(x)}, \qquad \frac{c'(x)}{p'(x)}$$

Keys to the solution

Lemma 2 (of ours)

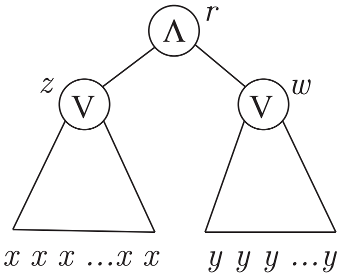
A certain constraint extremum problem has a unique solution.

The proof highlight: By means of Lemma 1, the objective function is decreasing in a certain open interval. □

Remark:
At the maximizer, the objective function is NOT differentiable.

Keys to the solution

Our constraint extremum problem:
Letters denote probabilities of having the value 0 at nodes.
$\boldsymbol{z}$ is the master variable. The objective function is the cost of the tree (the associated algorithm: left to right).
Side conditions: $\boldsymbol{r}$ is a probability constraint (fixed, $\mathbf{0} < \boldsymbol{r} < \mathbf{1}$).
$\mathbf{1} - \sqrt{\mathbf{1} - \boldsymbol{r}} < \boldsymbol{z} < \boldsymbol{r}$, $(\mathbf{1} - \boldsymbol{z})(\mathbf{1} - \boldsymbol{w}) = \mathbf{1} - \boldsymbol{r}$.



Our result

Theorem B (Our main result)

Fix an r $(0 < r < 1)$. Let rID denote an ID s.t. prob. of the root (having the value 0) is r.

If d achieves the equilibrium among rIDs then d is an IID.

As a corollary

Theorem A (Liu and Tanaka, 2007)

If d achieves the equilibrium among IDs then d is an IID.

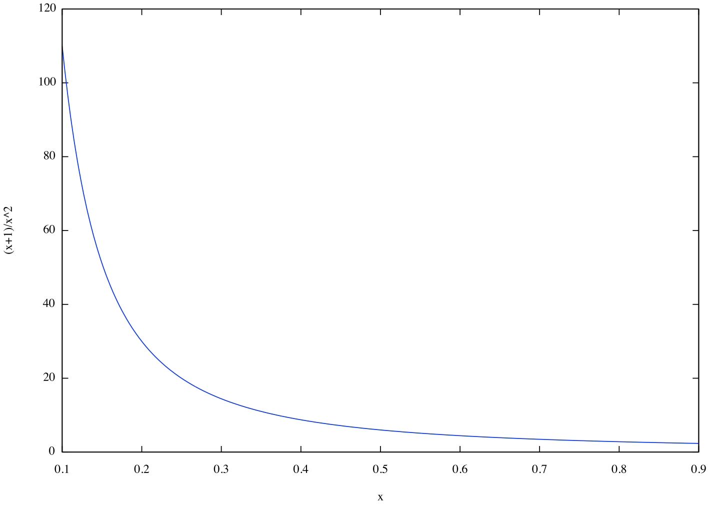


Figure 1: $\boldsymbol{c_{\vee,1}(x)/p_{\vee,1}(x)\ (0.1 < x < 0.9)}$

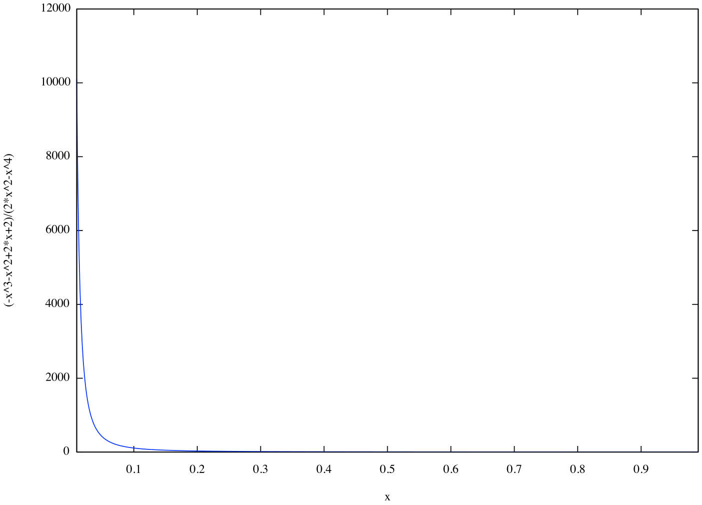


Figure 2: $c_{\vee,2}(x)/p_{\vee,2}(x)$ $(0 < x < 1)$

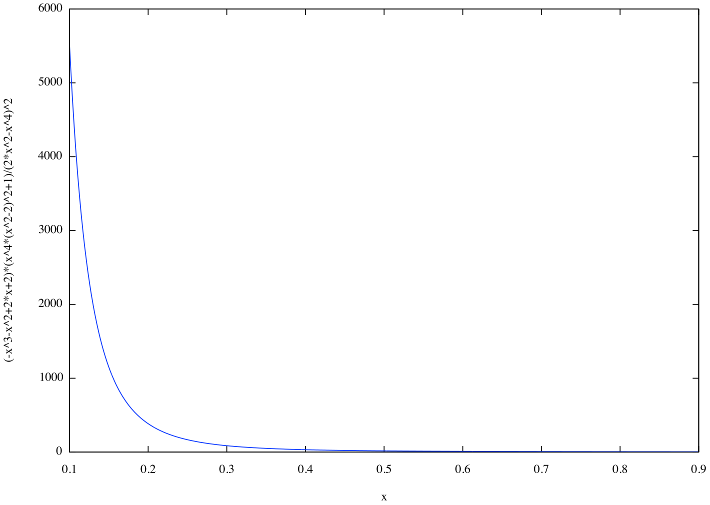


Figure 3: $c_{\vee,3}(x)/p_{\vee,3}(x)$ $(0.1 < x < 0.9)$

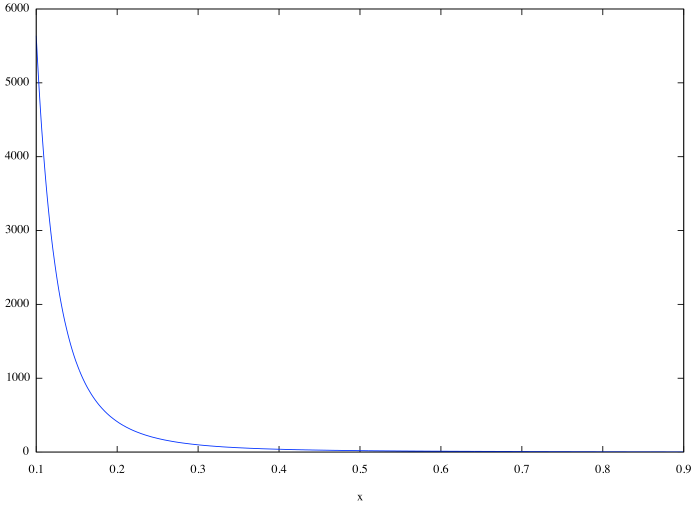


Figure 4: $c_{\vee,4}(x)/p_{\vee,4}(x)$ $(0.1 < x < 0.9)$

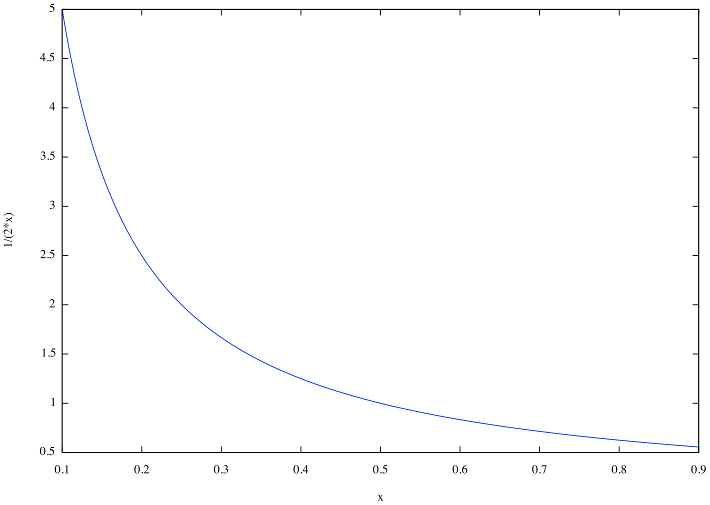


Figure 5: $c'_{\vee,1}(x)/p'_{\vee,1}(x)$ $(0.1 < x < 0.9)$

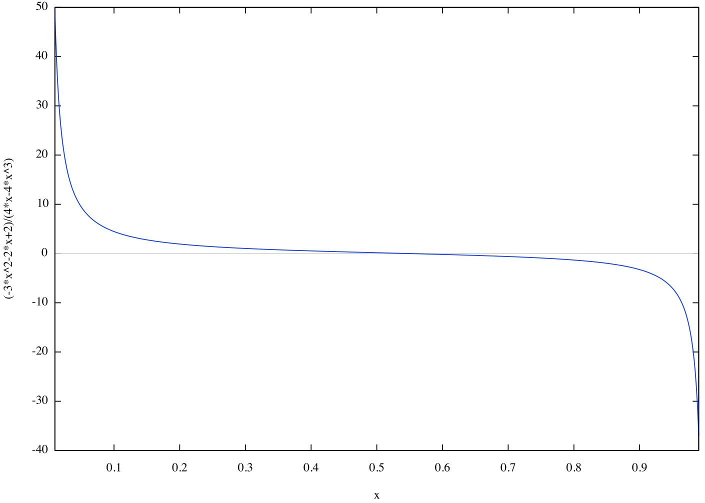


Figure 6: $c'_{\vee,2}(x)/p'_{\vee,2}(x)$ $(0 < x < 1)$

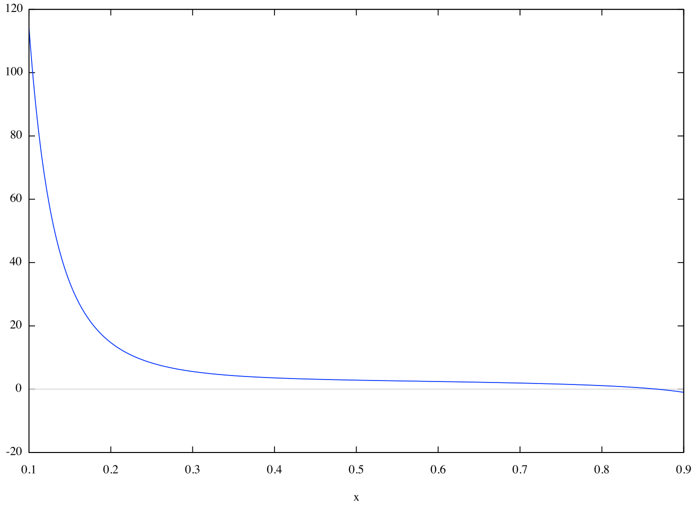


Figure 7: $c'_{\vee,3}(x)/p'_{\vee,3}(x)$ $(0.1 < x < 0.9)$

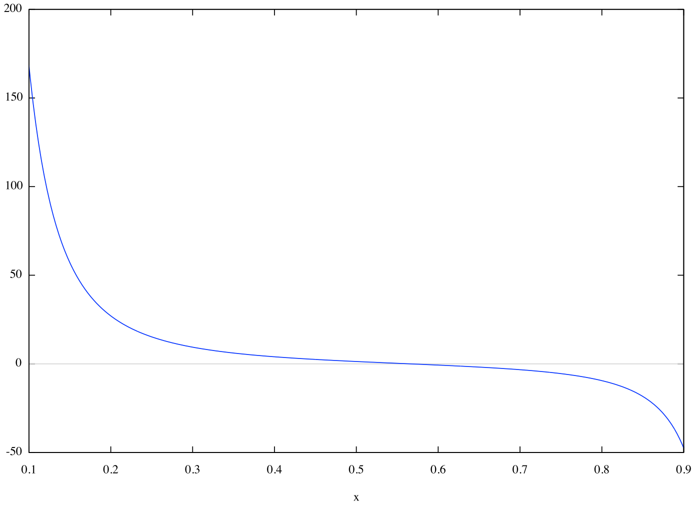


Figure 8: $c'_{\vee,4}(x)/p'_{\vee,4}(x)$ $(0.1 < x < 0.9)$

Thank you for your attention.

Happy 60th birthday.

arXiv:1401.8175

- Knuth, D.E. and Moore, R.W.: An analysis of alpha-beta pruning. *Artif. Intell.*, **6** pp. 293–326 (1975).
- Liu, C.-G. and Tanaka, K.: Eigen-distribution on random assignments for game trees. *Inform. Process. Lett.*, **104** pp.73–77 (2007).
- Saks, M. and Wigderson, A.: Probabilistic Boolean decision threes and the complexity of evaluating game trees. In: *Proc. 27th FOCS*, pp.29–38 (1986).
- Suzuki, T. and Nakamura, R.: The eigen distribution of an AND-OR tree under directional algorithms. *IAENG Internat. J. of Applied Math.*, **42**, pp.122-128 (2012). `www.iaeng.org/IJAM/issues_v42/issue_2/index.html`