Forcing Complexity

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Abstract

We overview our research on forcing complexity:

- **Forcing complexity** (of a given formula)
  - Min. size of forcing conditions (its domain) which force it.

- **A Dowd-generic set**
  - A set of strings s.t. associated query formulas have small forcing complexity (We will show precise def. later.)

- A resource-bounded generic set of Ambos-Spies et al.

- Connections with:
  - resource-bounded randomness
  - computational complexity
Outline

Abstract

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§1 Background

Def. Resource-bounded randomness (Ambos-Spies et al.)

\( t(n) \)-random
\( \simeq \) random for \( O(t(n)) \)-time computable martingales.

Two Types of Resource-bounded Genericity

- [Ambos-Spies and Mayordomo 1997],
  
  \( t(n) \)-random \( \Rightarrow \) \( t(n) \)-stochastic \( \Rightarrow \) \( t(n) \)-generic.

- [Dowd 1992]: Based on analogy of forcing theorem.
  [Kumabe and S.]: There exists an elementary recursive \( t(n) \) s.t.
  
  \( t(n) \)-random \( \Rightarrow \) Dowd-generic.
§1 Background

Dowd introduced the following in his study of NP=?coNP question.

Def. of Dowd-generic sets (sketch)

“A certain property* of an \textit{exponential-sized} portion of an oracle \(X\) is forced by a \textit{polynomial-sized} portion of \(X\).”

“A certain property” is described with \textit{the relativized propositional calculus (RPC)}.

\[
\text{RPC} = (\text{propositional calculus}) + \{\xi^1(\_), \xi^2(\_, \_), \xi^3(\_, \_, \_), \ldots\}
\]

For each \(n\), the \(n\)-ary connective \(\xi^n\) (a \textit{query symbol}) is interpreted to the initial segment of a given oracle up to \(2^n\)th string.
Example of a formula of RPC

\[(q_0 \Leftrightarrow \xi^3(q_1, q_2, q_3)) \Rightarrow [q_0 \lor (q_1 \land q_4)]\]

Given a formula $F$ of RPC and an oracle $X$, truth of $F$ is determined by “a truth assignment + a finite portion of $X$”.

Interpretation:

$\xi^n(i\text{th of } \{0, 1\}^n)$ is interpreted as to be $X(i\text{th of } \{0, 1\}^*)$,

where “$i\text{th}$” is that of length-lexicographic order.
§1 Background

$\xi^n(i\text{th of }\{0,1\}^n)$ is interpreted as to be $X(i\text{th of }\{0,1\}^*)$.

Examples ($n = 2$ and $n = 3$)

\[
\begin{array}{cccc}
\xi^2(0, 0) & \xi^2(0, 1) & \xi^2(1, 0) & \xi^2(1, 1) \\
X(\text{empty string}) & X(0) & X(1) & X(00)
\end{array}
\]

\[
\begin{array}{cccc}
\xi^3(0, 0, 0) & \xi^3(0, 0, 1) & \xi^3(0, 1, 0) & \xi^3(0, 1, 1) \\
X(\text{empty string}) & X(0) & X(1) & X(00)
\end{array}
\]

\[
\begin{array}{cccc}
\xi^3(1, 0, 0) & \xi^3(1, 0, 1) & \xi^3(1, 1, 0) & \xi^3(1, 1, 1) \\
X(01) & X(10) & X(11) & X(000)
\end{array}
\]

Thus, $\xi^2(q_2, q_1)$ and $\xi^3(0, q_2, q_1)$ are interpreted as to be the same.
§1 Background

Def. Force

A finite portion $\sigma$ (a finite sub-function) of an oracle $X$ is called a \textit{forcing condition}.

$\sigma$ \textit{forces} $F$ if for any $Y$ extending $\sigma$, $F$ is a tautology w. r. t. $Y$.

Example of force

Let $F$ be: $(q_0 \Leftrightarrow \xi^3(q_1, q_2, q_3)) \Rightarrow \neg q_0$

$F$ is a tautology w. r. t. the characteristic func. of the empty set. If $\sigma$ forces $F$ then the size of $\sigma$ (its domain) $\geq 2^3$. (And, the first $2^3$ bits of $\sigma$ should be 0.)
§2 Results on Non-Existence

The case of unbounded occurrences of query symbols

Definition. t-generic sets [Dowd 1992]

\( X \) is \textit{t-generic} if every tautology \( F \) with respect to \( X \) is forced by a forcing condition of polynomial-size in \(|F|\).

Thm. Non-existence of t-generic sets [Dowd, 1992], [S. 2001]

There are no t-generic sets.


3 Results on Existence

The case of **bounded** occurrences of query symbols:
Here, \( r \)-query denotes “the \( \# \) of occurrences of query symbols is \( r \).”

**Def. Dowd-generic sets [Dowd, 1992]**

- Let \( r \) be a positive integer.
  - \( X \) is **\( r \)-Dowd**
    - if every \( r \)-query tautology \( F \) w. r. t. \( X \)
    - is forced by a forcing condition of polynomial-size in \(|F|\).

- \( X \) is **Dowd-generic**
  - if \( X \) is \( r \)-Dowd for every positive integer \( r \).
  - (Polynomial bound depends on each \( r \), unlike \( t \)-genericity)
§3 Results on Existence

Thm. Existence of Dowd-generic sets

- [Dowd, 1992], [S.2001], [S.2002]
  The class of all Dowd-generic sets has Lebesgue measure 1.


S.: Degrees of Dowd-type generic oracles.

§3 Results on Existence

[Dowd 1992] asserts “any 1-Dowd set is not c.e.” (false)

**Thm. Degrees of Dowd-generic sets**


- [Kumabe and S. 2012] The same holds for “Dowd-generic” in place of “1-Dowd”.

§3 Results on Existence

Thm. Resource-bounded randomness implies Dowd-genericity

[Kumabe and S. ∞]
There exists an elementary recursive function $t(n)$ s.t.
$t(n)$-random $\Rightarrow$ Dowd-generic.

Gives an alt. proof: $\exists$ a primitive recursive Dowd-generic set.

Kumabe, M. and S. :
Resource-bounded martingales and computable Dowd-type generic sets. submitted to a journal (2010).
§3 Jump and a Problem

Let $1\text{TAUT}^X$ denote the set of all 1-query tautologies w. r. t. $X$.

Question: Does $1\text{TAUT}^X$ has a degree strictly higher than $X$?

Given a reduction concept $\leq_r$ (e.g., poly.-time Turing $\leq_T^P$), we introduce the following statement, and we call it “One-query jump hypothesis w. r. t. $\leq_r$” ($1\text{QJH}(r)$, for short).

**Def. One-query Jump Hypothesis w. r. t. $\leq_r$ [S. 2002]**

“The class $\{X : X <_r 1\text{TAUT}^X\}$ has Lebesgue measure 1 in the Cantor space”.
§3 Jump and a Problem

Thm. [S. 1998]

\[ 1QJH(\text{poly.-time Turing}) \iff \text{RP} \neq \text{NP}. \]

Here, RP is the one-sided version of BPP.

Thm. [S. 2002]

\[ 1QJH(\text{poly.-time truth table}) \implies \text{P} \neq \text{NP}. \]

S.: Recognizing tautology by a deterministic algorithm whose while-loop’s execution time is bounded by forcing.  
§3 Jump and a Problem

Examples of 1QJH [Kumabe, S. and Yamazaki 2008]

(1) 1QJH (monotone reductions) holds.
   (tt-reductions s.t. truth tables are monotone Boolean formulas.)

(2) \( c < 1 \Rightarrow 1QJH \) (tt-reductions s.t. norm \( \leq c \times |F| \)) holds.
   (\( F \) is an input formula and \( |F| = \# \) of occurrences of symbols.)

Problem

In (2), can we relax the assumption of “\( c < 1 \)”?

## Summary

### §1 Results on Non-Existence (Thm. [Dowd 1992], [S. 2001])

No t-generic sets (No poly.-bound on forcing complexity when unbounded occurrences of query symbols).

### §2 Results on Existence (Def.)

1. $r$-Dowd $\iff$ poly.bound on forcing comp. for $r$-query tautologies
2. Dowd-generic $\iff \forall r \geq 1 \ r$-Dowd

### §2 (Thm. [Kumabe and S., $\infty$])

There exists an elementary recursive function $t(n)$ s.t. $t(n)$-random $\Rightarrow$ Dowd-generic.
(Hence $\exists$ a primitive recursive Dowd-generic set.)
§3 Jump and a Problem (Def.)

1QJH(\(r\)): “\( \{ X : X < r, 1TAUT^X \} \) has Lebesgue measure 1”.

§3 (Thm. [Kumabe, S. and Yamazaki 2008])

(1) 1QJH(monotone reductions) holds.
(2) \( c < 1 \Rightarrow 1QJH(tt\text{-}reductions \text{ s.t. norm } \leq c \times |F|) \) holds. 

(\( F \) is an input formula and \( |F| = \# \) of occurrences of symbols.)

§3 (Problem)

In (2), can we relax the assumption of \( “c < 1” \)?
Thank you.
References


Suzuki, T.: Forcing complexity: minimum sizes of forcing conditions.

Suzuki, T.: Degrees of Dowd-type generic oracles.

Suzuki, T.: Bounded truth table does not reduce the one-query tautologies to a random oracle.


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