Forcing complexity: minimum sizes of forcing conditions

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Forcing complexity: minimum sizes of forcing conditions

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Abstract

This note is a continuation of our former paper "Complexity of the r-query tautologies in the presence of a generic oracle" (*Notre Dame J. Formal Logic* 41 (2000)). We give a very short direct proof of the non-existence of t-generic oracles, a result obtaind first by Dowd (*Inform. and Comput.* 96 (1992)). We also reconsitute a proof of Dowd's result that the class of all r-generic oracles in his sense has Lebesgue measure one.
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1 Introduction

In a series of our former papers [4, 6, 7], by extending Dowd's pioneering work [2], we studied complexity issues on minimum sizes of forcing conditions. This short note is a continuation of [6]. In [6], we produced a NP predicate, and by using it as a tool, we gave a (very short) proof of the fact that the class of t-generic oracles has measure zero; in this note, a better chosen NP predicate provides an equally short proof of a more drastic fact: this class is empty. The original proof of this last fact, by Dowd [2], was less direct; more over, in [6], we showed the original proof's logical gap by presenting a counter example. We also reconstitute a proof of the fact that for each positive integer r, the class of all r-generic oracles in the sense of Dowd has Lebesgue measure one. The original proof of this fact [2] was difficult to understand. The preliminary version of this note was cited in [5, 7] as "Forcing complexity: supplement to complexity of the r-query tautologies."

We refer to [6] and [7] for history, motivations, definitions and notations. We only state minimum of definitions here. In the sequel X is a symbol for an *oracle*, that is a set of bit strings, or better the characteristic function of such a set. A *forcing condition* means a restriction of an oracle to a finite domain. By "forcing complexity" we mean the minimum size of forcing conditions that force a given predicate. More formally, as follows. Let y be a variable for a bit string. Assume $\varphi(X, y)$ is an arithmetical predicate. Further, we assume φ is *finitely testable* (Poizat [3]), that is, there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that for every oracle A and every bit string u, $\varphi(A, u)$ holds if and only if $\varphi(B, u)$ holds, where B is the extension of $A \restriction (\{0,1\}^{\leq n})$ such that $B(u) = 0$ for all u such that $|u| > n$. Assume A is an oracle and n is a natural number. *The forcing complexity of* φ *relative to* A *at* n is the least natural number k of the following property: for any bit string u of length n, if $\varphi(A, u)$ is true then A has a finite portion S of size at most k such that S forces $\varphi(X, u)$: in other words, the cardinality of $\mathrm{dom}(S)$ is at most k and for any oracle B extending S, $\varphi(B, u)$ is true. An oracle is called *t-generic* if relative to which the forcing complexity of (the property of being) relativized tautologies is at most polynomial.

2 The non-existence of t-generic oracles

We define an arithmetical (in fact NP) predicate $\mathrm{CS}(X, y)$ as follows: "y is not the empty string and, letting $n = |y| - 1$, there exist strings $u_1, \ldots, u_{n+1}$ of length n, which are consecutive in the lexicographic ordering, such that $y = X(u_1) \cdots X(u_{n+1})$." And, $\mathrm{coCS}(X, y)$ is the negation of it. The letters CS are for "consecutive strings."

Theorem 1 *Assume that A is an oracle and n is a natural number. Then the forcing complexity of* coCS *relative to A at $n+1$ is at least* $(2^n - n)/(n+1)$.

Proof: Assume for a contradiction that the forcing complexity is less than $(2^n - n)/(n+1)$. Among bit strings of length $n+1$, at most 2^n bit strings v's make the assertion $\mathrm{CS}(A, v)$ true, so that we can find one of them, say u, for which $\mathrm{coCS}(A, u)$ is true. By hypothesis, $\mathrm{coCS}(X, u)$ (where only u is fixed) is forced by a finite portion S of A, whose domain has d elements, where $d < (2^n - n)/(n+1)$.

Since $n(d+1) < 2^n - d$, the set R of strings of length n which are not in the domain of S contains more than $n(d+1)$ elements. This set R being composed of a maximum of $d+1$ intervals for the lexicographic ordering, one of them contains at least $n+1$ elements. Therefore, we obtain an oracle B extending S and satisfying $\mathrm{CS}(B, u)$: a contradiction. □

In [6, §3], by using the the predicate CORANGE of Bennet and Gill [1], we presented a short proof of the fact that the class of all t-generic oracles has Lebesgue measure zero. By using coCS in place of CORANGE, we drastically improve it as follows.

Corollary 2 ([2, Lemma 7]) *t-generic oracles do not exist.*

Proof: According to [2, §3], a t-generic oracle should force any coNP predicate with the help of one of its polynomial-sized fragment, contradicting Theorem 1. □

3 Reconstitution: r-generic oracles

For a positive integer r, an oracle is *r-generic in the sense of Dowd* [2] if it satisfy the definition of a t-generic oracle for r-query tautologies in place of relativized tautologies. This r-genericity is completely different from that of arithemtical forcing. In the remaininig part of this note, r-genericity always means that of Dowd, though we do not think it is a good terminology (in [7], in order to avoid confusion, we used terminology *r-Dowd oracles* instead of r-generic oracles in the sense of Dowd). In this section, we reconstitute a proof of the following fact (**Dowd's Theorem 10**, [2]): *For every positive integer r, the class of all r-generic oracles has Lebesgue measure one in the Cantor space.*

Dowd proved his Theorem 10 by using the following fact (**Dowd's Lemma 9**, [2]): *If F is a 1-query tautology with respect to some oracle then there is a unique minimal forcing condition S that forces F (to be a tautology).* Here, the assumption of 1-query is critical. For example, if G is a 2-query formula asserting that exactly one of two strings 001 and 101 belongs to a given oracle then threre are two minimal forcing conditions that force G. Therefore, when $r \geq 2$, we cannot rely on the uniqueness of the minimal forcing condition that forces a given formula. However, Dowd's original proof of his Theorem 10 [2, p.70 line 33 – p.71 line 14] seems relying on the uniqueness. And, it is difficult to understand the structure of induction used there. In the author's doctoral dissertation [5, Chapter 4], a proof of Dowd's Theorem 10 was rigorously reconstituted, but it was long and complicated. Here, we sketch it without detail. There are two important ideas.

The first is a "partial version" of forcing complexity. Suppose $r \geq 2$. For each oracle A, we define a certain subset of r-query tautologies with respect to A; for the time being, we call them *nice r-query tautologies with respect to A*. They have the following property (a **revised version of Dowd's Lemma 9**): *If F is a nice r-query tautology with respect to some oracle X, and if T is a (certain special type of) finite portion of X, then there is a unique minimal forcing condition S_T such that the union of S_T and T forces F.*

The second important idea is oracles' hierarchy with respect to forcing complexity. Suppose $r \geq 2$. Let $r\mathrm{GEN}_1$ be the class of all oracles D such that for every nice r-query tautology F and every (certain type of) finite portion T of D, the S_T has polynomial size of $|F|$. Let $r\mathrm{GEN}_2$ be the class of all oracles for which the forcing complexity of (the property of being) nice r-query tautologies is at most polynomial. For each $r \geq 1$, let $r\mathrm{GEN}_3$ be the class of all r-generic oracles. We get the following scheme.

$$\begin{array}{ccccccc}
 & & 2\mathrm{GEN}_1 & \supseteq & 3\mathrm{GEN}_1 & \supseteq \cdots \\
 & & \cup\!\!| & & \cup\!\!| & \\
 & & 2\mathrm{GEN}_2 & \supseteq & 3\mathrm{GEN}_2 & \supseteq \cdots \\
 & & \cup\!\!| & & \cup\!\!| & \\
1\mathrm{GEN}_3 & \supseteq & 2\mathrm{GEN}_3 & \supseteq & 3\mathrm{GEN}_3 & \supseteq \cdots
\end{array}$$

Claim 1. *For each $r \geq 2$, $r\mathrm{GEN}_1$ has Lebesgue measure one.* Proof (sketch): Simalr to Dowd's proof of the existence of 1-generic oracles [2, p.70]. Instead of

Dowd's Lemma 9, we use the revised version of it. See also [7, §4].

Claim 2. *We have* $2\text{GEN}_1 \subseteq 1\text{GEN}_3$*.* Proof (sketch): This is shown by adding dummy symbols to a given 1-query formula.

Claim 3. *For each* $r \geq 1$*, we have* $r\text{GEN}_3 \cap (r+1)\text{GEN}_1 \subseteq (r+1)\text{GEN}_2$. Proof (sketch): Simalr to Dowd's original proof of his Theorem 10 for $r \geq 2$ [2, p.71, line 6–13]. We use the revised version of Dowd's Lemma 9.

Claim 4. *For each* $r \geq 1$*, we have* $r\text{GEN}_3 \cap (r+1)\text{GEN}_2 \subseteq (r+1)\text{GEN}_3$. Proof (sketch): An $(r+1)$-query tautology is equivalent to a certain "simultaneous equation" consisting of r-query tautologies and nice tautologies, where the number of these tautologies is at most polynomial, because r is fixed.

Now, we show that each vertical hierarchy collapses. By induction on r with Claim 2, 3 and 4, we have $(r+1)\text{GEN}_1 \subseteq r\text{GEN}_3$ and $(r+1)\text{GEN}_1 = (r+1)\text{GEN}_2 = (r+1)\text{GEN}_3$, for each $r \geq 1$. By this fact and Claim 1, for each $r \geq 1$, $r\text{GEN}_3$ has Lebesgue measure one. Thus, we have shown Dowd's Theorem 10.

A problem that we still leave open is whether the horizontal hierarchy collapses or not. See [7] for a partial answer to this problem.

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